Journal of Global Optimization **7**: 51–73, 1995.

A Recipe for Semidefinite Relaxation for (0,1)-Quadratic Programming

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(Received: 8 August 1994; accepted: 19 January 1995)

Abstract. We review various relaxations of (0,1)-quadratic programming problems. These include semidefinite programs, parametric trust region problems and concave quadratic maximization. All relaxations that we consider lead to efficiently solvable problems. The main contributions of the paper are the following. Using Lagrangian duality, we prove equivalence of the relaxations in a unified and simple way. Some of these equivalences have been known previously, but our approach leads to short and transparent proofs. Moreover we extend the approach to the case of equality constrained problems by taking the squared linear constraints into the objective function. We show how this technique can be applied to the Quadratic Assignment Problem, the Graph Partition Problem and the Max-Clique Problem. Finally we show our relaxation to be best possible among all quadratic majorants with zero trace.

Key words: Quadratic boolean programming, semidefinite programming, bounds, Lagrangian duality, parametric programming, trust region subproblems, minmax eigenvalue problems, quadratic assignment problem, graph partitioning, max-clique, theta function.

1. Introduction

1.1. Binary Quadratic Programming Problem

Consider the quadratic objective function

$$q(x) := x^t Q x - 2c^t x,$$

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* The research was partially supported by GAČR 201/93/0519.
** Research support by Christian Doppler Laboratorium für Diskrete Optimierung.
‡ Research support by the National Science and Engineering Research Council Canada.

where Q is an $n \times n$ symmetric matrix and $c \in \Re^n$. We study the general $(-1, +1)$-*quadratic programming problem*

$$(P) \quad \mu^* := \max\ q(x), \quad x \in F \cap S,$$

where $F = \{-1, 1\}^n$, $S \subset \Re^n$, and $F \cap S \neq \emptyset$. The (0,1)-quadratic programming problem is equivalent to (P) via the transformation $x = 2y - e$, where $y \in \{0, 1\}^n$ and e is the *vector of ones*.

Problem (P), without the extra complication given by the set S, is NP-hard. One approach to get efficiently solvable problems, often used in the literature, is to relax the constraints and perturb the objective function in order to obtain upper bounds. In this paper we study several relaxations for (P) with the additional restriction provided by the set S. Moreover, we study the possible perturbations, in the data, that need to be considered to maintain tractable relaxations. Our main result is that all the bounds we consider can be obtained from the Lagrangian dual relaxation of an appropriate quadratically constrained equivalent program. This is done by adding appropriate redundant constraints and exploiting the hidden semidefinite constraints that occur in quadratic programming. Moreover, up to a normalization, the Lagrangian approach provides all the correct perturbations needed. Therefore, the Lagrangian dual is shown to be a very powerful tool, and semidefinite relaxations are shown to be equivalent to quadratic relaxations. This takes the guesswork out of forming semidefinite relaxations and shows which ones are guaranteed to be equivalent even though they may appear very different. All of the bounds that we discuss are tractable, i.e., can be found in polynomial time. This includes the trust region bounds and the box constrained bound. Therefore we have not added comments to this effect.

An illustration of (P) is the *Quadratic Assignment Problem* (in the trace formulation), see, e.g. [8],

$$(QAP) \quad \max_{X \in \Pi}\ q(X) := \operatorname{trace}(AXB - 2C)X^t,$$

where Π denotes the set of $n \times n$ permutation matrices, A, B are symmetric $n \times n$ matrices, and C is an $n \times n$ matrix. In this case, the feasible set (of matrices) is the set (F translated) of (0,1)-matrices intersected with the set (S translated) of matrices whose row and column sums are all 1. This shows the usefulness of the set S in (P). (This set is used in branch and bound applications as well.)

Another example, and in fact an equivalent problem when S is the whole space, is the *Max-Cut Problem*. Let $G = (V, E)$ be an undirected graph with edge set $V = \{v_i\}_{i=1}^n$ and weights w_{ij} on the edges $(v_i, v_j) \in E$. We want to find the index set $\mathcal{I} \subset \{1, 2, \ldots n\}$, to maximize the weight of the edges with one end point with index in $\mathcal{I}$ and the other in the complement. This is equivalent to

$$(MC) \quad \max\ \tfrac{1}{2} \textstyle\sum_{i<j} w_{ij}(1 - x_i x_j), \quad x \in F,$$

where $x_i = 1$ if $i \in \mathcal{I}$ and -1 otherwise. This is an instance of (P) with a pure quadratic objective function. The corresponding matrix Q has components

$q_{ij} = -w_{ij}$, with 0 diagonal. Alternatively, Q can be taken as the Laplacian matrix of the graph, see, e.g. [6]. The resulting eigenvalue bound and the equivalent semidefinite bound have been recently used in numerical studies and found to do exceptionally well, e.g. [13]. Moreover, the semidefinite bound has been studied in [6] and shown to have a particular good performance index, see [10].

Other instances that we consider include: graph partitioning and max-clique problems.

1.2. Historical Background

Quadratic bounds using a Lagrangian relaxation have been extensively studied and applied in the literature, for example in [16] and, more recently, in [17]. The latter calls the Lagrangian relaxation the "best convex bound". Discussions on Lagrangian relaxation for nonconvex programs also appear in [9]. In [23] it was shown that several quadratic type bounds considered in the literature, and in particular the ones mentioned in the abstract, are actually equal. A similar phenomenon occurs for linearizations of (P), such as in *roof duality*, see, e.g. [12], where many bounds obtained from various linearizations have been shown to be equal and, in fact, they have been shown to be equal to the Lagrangian dual of a linearized problem, see [1].

Semidefinite relaxations have recently appeared in relation to relaxations for 0–1 optimization problems. In [19], a "lifting" procedure is presented to obtain a problem in $\Re^{n^2}$; and then the problem is projected back to obtain tighter inequalities. See also [3]. Several of the operators that arise in our applications are similar to those that appear in [19]. However, our motivation and approach is different. A discussion of several applications for semidefinite relaxation appears in [2].

1.3. Outline

This paper is organized as follows. After several preliminary notions and definitions, we study (P) in the simple case that $S = \Re^n$. This prepares the way for the general case and provides intuition and motivation for deciding which relaxations and which parametrizations are important. In Section 2.1 we present several relaxations for (P). These include: the trivial relaxation of ignoring the constraint set F; relaxing F to the sphere and then to the box; homogenization by moving to a larger dimensional space; and *lifting* to the space of semidefinite matrices. Many of these relaxations are well known in the literature. The purpose of presenting them is to show, in Section 2.2, that all these different looking bounds are actually obtained from, and equal to, the Lagrangian dual of a quadratically constrained equivalent problem to (P). Moreover, the Lagrange multipliers provide the perturbations used in these bounds. Then, in Section 2.3 we show that the perturbations that arise from the Lagrange multipliers provide all the quadratic majorants with zero trace.

We then continue in Section 3 with S defined by linear equality constraints. The development is very similar to the one for the simple $S = \Re^n$ case. We show that the correct hidden semidefinite constraint provides the correct way of introducing linear constraints to the objective function. This is done by squaring the constraints. This provides a recipe for obtaining a quadratic relaxation, as well as the semidefinite relaxation, for hard combinatorial problems.

We then present several specific applications in Section 4. The quadratic assignment problem, QAP, is treated in Section 4.1. The graph partitioning problem is treated in 4.2. Finally, the max-clique problem appears in 4.3.

1.4. PRELIMINARIES

We will use the following notations: Diag (v) denotes the *diagonal matrix* formed from the vector v, while the *adjoint operator*, Diag $^*(M) =$ diag (M), is the vector of the diagonal elements of the matrix M; $\mathcal{R}(M), \mathcal{N}(M)$ denote the *range space* and *null space*, respectively; e is the vector of ones and e_i is the i-th unit vector; for symmetric matrices $M_1 \preceq M_2$ $(M_1 \prec M_2)$ refers to the Löwner partial order, i.e., that $M_1 - M_2$ is negative semidefinite (negative definite, respectively); similar definitions hold for positive semidefinite and positive definite; $v \leq w$, $(v < w)$ refers to coordinatewise ordering of the vectors; conv (S) is the convex hull of the set S; $\lambda_{\max}(M)$ $(\lambda_{\min}(M))$ denotes the largest (smallest) eigenvalue of a symmetric matrix M. The *space of $n \times n$ symmetric matrices*, denoted $\mathcal{S}_n$, is considered with the trace inner product $< M, N >=$ trace MN.

We use the *Kronecker product*, or tensor product, of two matrices, $A \otimes B$, formed from the matrix blocks $A_{ij}B$, when discussing the quadratic assignment problem QAP; vec X denotes the vector formed from the columns of the matrix X, while Mat x denotes the matrix formed from the vector x. The *Hadamard product* or elementwise product, is denoted $A \circ B$. The Kronecker product gives rise to generalized notions of trace and diagonal. For a $(n^2 + 1) \times (n^2 + 1)$ matrix Y, $\mathrm{b}^0\mathrm{diag}\,(Y)$ is an $n \times n$ matrix called the *block diagonal sum* of Y. It is formed by ignoring the first row and column of Y and then summing the next n blocks of $n \times n$ principal submatrices. The adjoint operator is denoted $\mathrm{B}^0\mathrm{Diag}\,(S)$, i.e. for the $n \times n$ matrix S, we get a block diagonal $(n^2 + 1) \times (n^2 + 1)$ matrix with the first row and column 0 and n blocks of the matrix S on the diagonal.

2. Special Case that $S = \Re^n$

In this section we first consider the special case when S is the whole space. The problem (P) is then equivalent to the max-cut problem. This section prepares the way for the general problem (P). We see that the Lagrangian dual provides a way of finding semidefinite relaxations as well as finding the proper parameters, or perturbations, to include in the relaxations. In fact, the Lagrange multipliers are exactly the correct perturbation parameters.

2.1. RELAXATIONS

The problem (P) is NP-hard. The common approach is to solve relaxations and obtain bounds which can be incorporated in a branch and bound routine. We now derive several new and well known bounds. The motivation for these bounds varies. For example, one bound relaxes the constraints to the unit ball of length n, while another relaxes the constraints to the convex hull, i.e., to the unit cube. Our motivation is to show that that all these bounds are actually equal to the one obtained from a Lagrangian relaxation.

We first note that, perturbing the diagonal of Q does not change the objective function q on the feasible set F if, in addition, we subtract the sum of the perturbations, i.e.

$$\begin{aligned} q_u(x) &:= x^t(Q + \mathrm{Diag}\,(u))x - 2c^t x - u^t e \\ &= q(x), \quad \forall x \in F. \end{aligned} \tag{2.1}$$

Let us define the trivial bound obtained from ignoring the constraints and allowing the diagonal perturbations. We define

$$f_0(u) := \max_x q_u(x). \tag{2.2}$$

This function can take on the value $+\infty$. We then get the following trivial bound.

$$\mu^* \leq B_0 := \min_{u^t e=0} f_0(u) = \min_u f_0(u). \tag{2.3}$$

We now see that this bound is equivalent to several others as well as to the smallest quadratic bound. (This extends the equivalences presented in [23].)

We can minimize over the unconstrained parameter u or add the restriction to $u^t e = 0$. This can be seen from the optimality conditions for min-max problems. (Details can be found in [23].) In addition we can restrict the parameters and avoid infinite values for the inner maximization problem by adding the hidden semidefinite constraint, i.e., we use the fact that a quadratic function is unbounded if the Hessian is indefinite. (Note that a quadratic function is bounded above if and only if the Hessian is negative semidefinite and the stationarity equation is consistent.)

$$\mu^* \leq B_0 = \min_{Q+\mathrm{Diag}\,(u)\preceq 0} f_0(u). \tag{2.4}$$

The feasible set for (P) lies on the sphere of radius $\sqrt{n}$. Then one relaxation of (P) is

$$f_1(u) := \max_{||x||^2=n} q_u(x). \tag{2.5}$$

This yields our next bound

$$\mu^* \leq B_1 := \min_{u^t e=0} f_1(u) = \min_u f_1(u). \tag{2.6}$$

The inner maximization problem is called a trust region subproblem, see e.g. [21]. This problem provides an important technique in unconstrained minimization. This bound will provide the central tool in our analysis.

We can replace the spherical constraint with the box constraint.

$$f_2(u) := \max_{|x_i|\leq 1} q_u(x). \tag{2.7}$$

After adding the semidefinite constraint to make the bound tractable, i.e. to make the calculation of f_2 tractable, we get our next bounds.

$$\mu^* \leq B_2 := \min_{u^t e=0} f_2(u) = \min_u f_2(u) \tag{2.8}$$

and

$$\mu^* \leq B_2 = \min_{Q+\mathrm{Diag}(u)\preceq 0} f_2(u). \tag{2.9}$$

Given Q and c, define the $(n+1)\times(n+1)$-matrix Q^c by adding a 0-th row and column, so that

$$\begin{array}{lll} q_{00}^c = 0 & & \\ q_{0i}^c = q_{i0}^c = -c_i & \text{for } i > 0 \\ q_{ij}^c = q_{ij} & \text{for } i, j > 0, \end{array}$$

i.e.

$$Q^c := \begin{bmatrix} 0 & -c^t \\ -c & Q \end{bmatrix}. \tag{2.10}$$

In order to have analogous functions $q_u^c(y)$ and $f_i(u)$ as in the previous cases, let us introduce

$$q_u^c(y) := y^t(Q^c + \mathrm{diag}\,(u))y - u^t e. \tag{2.11}$$

Note that q_u^c reduces to q_u if the first component y_0 is ± 1. The equivalent relaxed problem is

$$f_1^c(u) := \max_{||y||^2=n+1} q_u^c(y) = (n+1)\lambda_{\max}(Q^c + \mathrm{diag}\,(u)) - u^t e. \tag{2.12}$$

Just as for f_1, we can restrict $u^t e = 0$. Now a bound for (P) is

$$B_1^c := \min_{u^t e=0} f_1^c(u) = \min_u f_1^c(u). \tag{2.13}$$

Similarly, we get equivalent bounds B_0^c and homogenized bounds for the other models.

The above argument shows that we can homogenize the problem by moving into a higher dimension. Therefore, we can consider the special case that $c = 0$.

TABLE I. Bounds for (P)

B_0	$=$	$\min_u \max_x q_u(x)$
B_1	$=$	$\min_u \max_{x^t x=n} q_u(x)$
B_2	$=$	$\min_u \max_{-1\leq x_i \leq 1} q_u(x)$
B_3	$=$	$\max\{\text{trace}\, Q^c Y : \text{diag}\,(Y) = e,\ Y \succeq 0.\}$
B_1^c	$=$	$\min_u \max_{y^t y=n+1} q_u^c(y)$ and other homogenized bounds.

Semidefinite programming has been applied to provide relaxations for combinatorial problems, e.g. [18]. (See [2] for several examples.) In the case that $c = 0$, the relaxation to (P) is studied in [6] and shown to have a particular good performance index, see [10]. The relaxation comes from the fact that

$$x^tQx = \text{trace}\, x^tQx = \text{trace}\, Qxx^t$$

and, for $x \in F$, $y_{ij} = x_i x_j$ defines a symmetric, rank one, positive semidefinite matrix Y with diagonal elements 1. Therefore, we can *lift* the problem into the higher dimensional space of symmetric matrices. This yields the following relaxation and our bound 3.

$$\begin{array}{rcc} B_3 := & \max & \text{trace}\, QY \\ & \text{subject to} & \text{diag}\,(Y) = e \\ & & Y \succeq 0. \end{array}$$

(Note that if Y is restricted to rank-one matrices, then the relaxation is equivalent to (P).)

In Table I, we now summarize the upper bounds for (P) in the case that S is the whole space. We use the definition of the perturbed function q_u but do not include the fact that we could add the restriction $u^te = 0$, and/or the hidden semidefinite constraint, in all the bounds that use the perturbation.

2.2. Lagrangian Duality

We continue to assume the simple case that S is the whole space. We now show that all the above relaxations and bounds come from the Lagrangian dual of the following equivalent problem to (P)

$$\begin{array}{lcl} (P_E) & \max & q(x) = x^tQx - 2c^tx \\ & \text{subject to} & x_i^2 = 1,\ \ i = 1, \cdots, n. \end{array} \tag{2.14}$$

Note that the Lagrangian dual of (P_E) yields precisely our trivial first bound B_0 in (2.3). Our strategy is to add redundant constraints to (2.14) and exploit the fact that there is a hidden semidefinite constraint in the Lagrangian dual for problems where the Lagrangian is a quadratic function.

Our analysis is based on the fact that the trust region subproblem,

$$(TR)\ \ tr^* := \max_{x^t x = n} q(x),$$

has a zero duality gap. More precisely, define the Lagrangian for the trust region subproblem with $u = 0$

$$L(x, \lambda) = x^t Q x - 2c^t x + \lambda(x^t x - n). \tag{2.15}$$

THEOREM 1. *With the Lagrangian for the trust region subproblem defined in (2.15), we get*

$$\max_x \min_\lambda L(x, \lambda) = \min_\lambda \max_x L(x, \lambda). \tag{2.16}$$

The left-hand side of (2.16) is equivalent to the trust region subproblem (TR) while the right-hand side is the Lagrangian dual problem. This result is proved in [26] in greater generality, i.e., allowing an indefinite, or nonconvex, trust region constraint $\beta \leq x^t C x \leq \alpha$, with general $C = C^t$. A simpler proof is one that exploits the fact that the minimization problem in the right-hand side has the hidden semidefinite constraint that $Q + \lambda C \preceq 0$, i.e., the inner max problem is unbounded otherwise, see [28].

Let us now add the redundant constraint $x^t x = n$ to (2.14). The Lagrangian dual is then equal to bound B_1 by Theorem 1, i.e.

$$\min_{u,\lambda} \max_x q_u(x) + \lambda(x^t x - n) = \min_u \max_{x^t x = n} q_u(x) = B_1.$$

But we can equate $u + \lambda e = v$ and see that the left-hand side is equal to the trivial bound B_0 as well, i.e. we have shown that

$$B_0 = B_1. \tag{2.17}$$

It is interesting to note that the trivial bound makes sense even in the case that $c = 0$, i.e., the hidden semidefinite bound results in a min-max eigenvalue problem with optimal value $-u^t e$.

Now consider adding the redundant constraints $x_i^2 \leq 1$. These constraints describe the box constraints $-1 \leq x_i \leq 1$. They have associated Lagrange multipliers $\lambda = (\lambda_i)$, since Slater's constraint qualification holds at the origin. We see that

$$\begin{aligned} B_2 &= \min_u \max_{x_i^2 \leq 1} q_u(x) \\ &= \min_u \min_{\lambda \geq 0} \max_x q_u(x) + \sum_{i=1}^n \lambda_i (1 - x_i^2) \\ &= \min_u \max_x q_u(x) \\ &= B_0, \end{aligned}$$

since we can let the new multipliers λ be absorbed into the vector u. Thus

$$B_0 = B_2. \tag{2.18}$$

The same argument holds if we add the hidden semidefinite constraints first.

Now consider the Lagrangian of the homogeneous problem in $\Re^{n+1}$. We move the quadratic constraints in and out of the Lagrangian using Theorem 1. Here $v, y \in \Re^{n+1}$ $u, x \in \Re^n$ and we equate y with x_0, x and v with u_0, u. Equality of the bounds B_0^c, B_1^c, B_2^c with each other follows from the above arguments.

$$\begin{aligned} B_1^c &= \min_v \max_{y^t y = n+1} q_v^c(y) = \min_v \max_y q_v^c(y) \\ &= \min_{u, u_0} \max_{x, x_0} u_0(x_0^2 - 1) + x^t(Q + \mathrm{Diag}\,(u))x - 2x_0 c^t x - u^t e \\ &= \min_u \max_{x, x_0^2 = 1} x^t(Q + \mathrm{Diag}\,(u))x - 2x_0 c^t x - u^t e \\ &= B_0, \end{aligned}$$

for if $x_0 = -1$, then the sign of x changes.

Finally consider the bound B_3 in the special case that $c = 0$. The generalized Slater's constraint qualification holds for this program and so the dual can be found from

$$\min_y \max_{Y \succeq 0} \mathrm{trace}\,(QY) - y^t(\mathrm{diag}\,(Y) - e),$$

see, e.g. [27]. This can be rewritten as

$$\min_y \max_{Y \succeq 0} \mathrm{trace}\,\left((Q - \mathrm{Diag}\,(y))Y\right) + y^t e.$$

The inner maximization yields the value ∞ unless $Q - \mathrm{Diag}\,(y) \preceq 0$. This implies that the maximum is attained at $Y = 0$. Therefore the problem reduces to

$$\begin{array}{ll} \text{minimize} & y^t e \\ \text{subject to} & Q - \mathrm{Diag}\,(y) \preceq 0. \end{array}$$

We can shift the semidefinite constraint to get

$$Q - \mathrm{Diag}\,(y - \frac{e^t y}{n} e) \preceq \frac{e^t y}{n} I.$$

We can then use the substitution $w = y - \frac{e^t y}{n} e$ and $z = \frac{e^t y}{n}$ to get the equivalent program

$$\begin{array}{ll} \text{minimize} & nz \\ \text{subject to} & Q - \mathrm{Diag}\,(w) \preceq zI \\ & w^t e = 0. \end{array}$$

This last program is equivalent to the min-max eigenvalue problem or bound B_1. This proves the following theorem.

THEOREM 2. *All the bounds discussed above (see Table I) are equal to the optimal value of the Lagrangian dual of the equivalent program* (P_E).

Therefore, these seemingly unrelated relaxations yield the same bound. This is obtained from using semidefinite duality. Moreover, as seen by experience with the max-cut problem, the dual pair of programs yields an efficient primal-dual semidefinite programming algorithm. In the sequel we discuss what happens when the constraint $x \in S$ is included, but first we study which perturbations are important for the relaxations.

2.3. PERTURBATIONS

We now return to the $(-1,1)$-quadratic program (P) but still assume that S is the whole space. The parameters in the above relaxations were restricted to perturbations in the diagonal of the matrix, i.e., we considered the perturbed quadratic functions

$$q_u(x) = x^t(Q + \mathrm{Diag}\,(u))x - 2c^t x - u^t e.$$

These perturbations did not change the value of the function on the feasible set F. Moreover, exactly these perturbations arose repeatedly from the Lagrangian duals of appropriately chosen programs. A natural question arises whether we can improve the bounds by allowing more general perturbations in the matrix or whether the Lagrange multipliers provide the correct set of perturbations. Note that we can add a multiple of the identity to Q in order to add a constant to the function. Therefore, no constant is included in the definition (2.19). Moreover, if we knew the value of the error in the bound, then we could always subtract this constant and get a perfect bound. Therefore, we assume that the perturbation in the matrix Q satisfies $\mathrm{trace}\,(U) = 0$. The case $\mathrm{trace}\,(U) \neq 0$ is much harder to analyse, and is omitted here.

For U an $n \times n$ symmetric matrix and $d \in \Re^n$, define the general perturbation in the data

$$q_{U,d}(x) = x^t(Q + U)x + (d - 2c)^t x. \tag{2.19}$$

We now show that the diagonal perturbations are the most general ones that we need to consider under the zero trace normalization.

The following observation will turn out to be useful.

LEMMA 3. *Let $U = (u_{ij})$ be an $n \times n$ matrix. Then*

$$\sum_{x \in F} x^t U x = 2^n \mathrm{trace}\,(U).$$

Proof.

$$\sum_{x \in F} x^t U x = \sum_{x \in F} \sum_{i} x_i^2 u_{ii} + \sum_{x \in F} \sum_{i \neq j} x_i x_j u_{ij}$$

$$= 2^n \text{trace}\,(U) + \sum_{i \neq j} u_{ij} \left(\sum_{x_i = x_j, x \in F} 1 - \sum_{x_i \neq x_j, x \in F} 1 \right)$$
$$= 2^n \text{trace}\,(U) + \sum_{i \neq j} u_{ij} (2^{n-1} - 2^{n-1}). \qquad \square$$

LEMMA 4. *Suppose that U is an $n \times n$ symmetric matrix. Then the following are equivalent:*

$$\text{trace}\, U = 0 \textit{ and } x^t U x \geq 0, \quad \forall x \in F; \tag{2.20}$$

$$x^t U x = 0, \quad \forall x \in F; \tag{2.21}$$

$$U \textit{ is diagonal, and } \text{trace}\, U = 0. \tag{2.22}$$

Proof. Suppose (2.20) holds. Using Lemma 3, we conclude

$$0 \leq \sum_{x \in F} x^t U x = 2^n \text{trace}\,(U) = 0,$$

thus (2.21) holds.

Next suppose (2.21) holds. Using the lemma again we note that

$$0 = \sum_{x \in F} x^t U x = 2^n \text{trace}\,(U),$$

thus trace $(U) = 0$. Now let $i \neq j$ be fixed. We have

$$0 = \sum_{x \in F, x_i = x_j} x^t U x = 2^{n-1} u_{ij},$$

showing that U is diagonal as well, and (2.22) follows.

Finally, (2.22) trivially implies (2.20). $\square$

THEOREM 5. *Suppose that $q_{U,d}$ is defined as in (2.19). Then the following are equivalent:*

$$\text{trace}\, U = 0, \textit{ and } q_{U,d}(x) \geq q(x), \quad \forall x \in F \tag{2.23}$$
$$q_{U,d}(x) = q(x), \quad \forall x \in F; \tag{2.24}$$
$$U \textit{ is diagonal, } \text{trace}\, U = 0, \textit{ and } d = 0. \tag{2.25}$$

Proof. Without loss of generality we assume $d = 0$. Otherwise we homogenize by (2.10). Next note that in this case

$$q_{U,d}(x) - q(x) = x^t U x, \quad \forall x \in F.$$

Therefore the theorem is implied by Lemma 4. $\square$

REMARK 6. The above perturbation results translate naturally to the (0,1)-quadratic program. This can be seen by by using the substitution $\frac{1}{2}(y+e)$ for the (0,1)-variables and expanding the quadratic function q. The only perturbations of q that are needed are of the form

$$w_i x_i^2 - w_i x_i,$$

i.e., we perturb the diagonal of Q as before but then add a linear perturbation as well. This perturbation is exactly the one that arises by using the Lagrangian relaxation of $x_i^2 - x_i = 0$. This connection has also been observed in [4].

REMARK 7. If we fix certain components of the variable x at $+1$ or -1, then the above results can be applied to the principal submatrix corresponding to the components that are not fixed. Therefore, in a branch and bound algorithm framework, we need only consider diagonal perturbations of this principal submatrix. Note that if the components in the subset S are free, while those in the complement are fixed, and U is the perturbation matrix as above, then

$$0 = x^t U x = x_S^t U_S x_S + \sum_{i \in S} x_i \left(\sum_{j \notin S} U_{i,j} x_j \right) + x_{S^c}^t U_{S^c} x_{S^c},$$

where S^c denotes the complementary index set. The last term is a constant and can be assumed to be 0 by adding a multiple of the identity to U. The middle term has 0 contribution since the components in S can be multiplied by -1, as seen in the proof of the above theorem. So we can conclude from the above theorem that the off diagonal terms of U_S are 0.

3. General Case (P)

We now consider the general problem (P) where the constraint set S is defined by linear equality constraints, i.e.

$$S = \{x \in \Re^n : Ax = b\},$$

for some $m \times n$ matrix A and $b \in \Re^m$. Our approach is to replace the linear constraint by the squared norm constraint to get the following equivalent problem to (P).

$$\begin{array}{rll} \mu^* = & \max & q(x) = x^t Q x - 2c^t x \\ & \text{subject to} & \|Ax - b\|^2 = 0 \\ & & x_i^2 = 1, \quad \forall i. \end{array} \tag{3.26}$$

In this section we provide motivation for using the Lagrangian relaxation for the above quadratic program. The semidefinite relaxation is obtained as the dual of the Lagrangian relaxation. This provides a *recipe* for finding a relaxation and a primal-dual pair of programs for an interior point algorithm.

RECIPE

1. Replace (P) by the quadratic constrained program where the linear constraints are replaced by the norm squared constraint and the ± 1 (0,1 respectively) constraint is replaced by the squared variables being equal to 1 ($x_i^2 - x_i = 0$, respectively).
2. Take the Lagrangian dual of the quadratic constrained program to obtain a min-max problem of the Lagrangian.
3. Homogenize the Lagrangian.
4. Use the hidden semidefinite constraint to obtain a minimization semidefinite program.
5. Take the Lagrangian dual of the resulting semidefinite program to obtain the maximization semidefinite relaxation of the original program (P).

3.1. LAGRANGIAN RELAXATION

Several difficulties arise from introducing linear equality constraints. One approach is to find a matrix Z such that $\mathcal{R}(Z) = \mathcal{N}(A)$ and then eliminate the equality constraints by substituting $x = Zy$. However, this can result in a complete loss of the easy combinatorial structure such as the ± 1 constraint.

Another approach is to bring the equality constraints into the objective function using Lagrange multipliers, i.e. the new parametrized objective would be $q_\lambda(x) = q(x) + \lambda^t(Ax - b)$. However, the hidden semidefinite constraint is now $\nabla^2 q \preceq 0$, whereas the true semidefinite constraint should clearly be that the projected Hessian $Z^t \nabla^2 q Z \preceq 0$, or equivalently that $\nabla^2 q$ is negative semidefinite on the null space of A. This can create a duality gap in the Lagrangian relaxation.

EXAMPLE 3.1. Consider the simple quadratic problem

$$1 = \max\{-x_1^2 + x_2^2 : x_2 = 1\}.$$

The Lagrangian dual yields

$$1 < \infty = \min_{\lambda} \max_{x} -x_1^2 + x_2^2 + \lambda(x_2 - 1),$$

and shows that there is a duality gap. However, if we replace $x_2 = 1$ with $(x_2 - 1)^2 = 0$, then we have a zero duality gap, i.e.

$$1 = \max\{-x_1^2 + x_2^2 : (x_2 - 1)^2 = 0\},$$

and

$$\begin{aligned} \inf_{\lambda} \max_{x} -x_1^2 + x_2^2 + \lambda(x_2 - 1)^2 &= \inf_{\lambda \leq -1} \max_{x} -x_1^2 + x_2^2 + \lambda(x_2 - 1)^2 \\ &= \inf_{\lambda < -1} \frac{\lambda}{1 + \lambda} \\ &= 1. \end{aligned}$$

A similar example can be constructed with an additional ball constraint, if we use a pure quadratic objective function which is negative semidefinite and we add a trivial linear constraint $Ax = 0$, where A is nonsingular.

It is interesting that the difficulty disappears if we use the quadratic form of the linear constraints $||Ax - b||^2 = 0$. The following shows that changing to the quadratic constraint provides the correct hidden semidefinite constraint.

PROPOSITION 8. *Suppose that A is an $m \times n$ matrix and $b \in \Re^n$. Define*

$$q_\alpha(x) = q(x) - \alpha(Ax - b)^t(Ax - b).$$

Then the following are equivalent:

$$\begin{array}{l} \nabla^2 q_\alpha \prec 0, \textit{ for some } \alpha \in \Re \\ \nabla^2 q_\alpha \prec 0, \textit{ for sufficiently large } \alpha \in \Re \\ \nabla^2 q \prec 0, \textit{ on } \mathcal{N}(A). \end{array}$$

Proof. The proof of this well known result can be found in several books and is essential in, e.g., the theory of augmented Lagrangians. (See, e.g. [5, 20]). □

The above result shows that we can get the correct hidden semidefinite conditions by transforming the linear constraints to quadratic constraints. This provides further motivation and a possible explanation for the recent success of the lifting process into semidefinite programming. Moreover, we now see that lifting such a quadratic constraint, or trust region constraint, does not create a duality gap. (This lifting is actually equivalent to the lifting in the semidefinite relaxation, since the dual is actually the semidefinite relaxation.)

THEOREM 9. *Let $K \subset \Re^n$ be a finite set and let A and b be as in Proposition 8. Then there exists $\bar{\lambda} \in \Re$ such that*

$$\begin{aligned} \max\{q(x) : x \in K,\ ||Ax - b||^2 = 0\} &= \max\{q(x) : x \in K,\ ||Ax - b||^2 \le 0\} \\ &= \min_{\lambda \ge 0} \max_{x \in K} q(x) - \lambda ||Ax - b||^2 \\ &= \min_{\lambda} \max_{x \in K} q(x) - \lambda ||Ax - b||^2 \\ &= \max_{x \in K} q(x) - \lambda ||Ax - b||^2,\ \forall \lambda \ge \bar{\lambda}. \end{aligned}$$

Proof. Define the function

$$h(\lambda) := \max_{x \in K} q(x) - \lambda ||Ax - b||^2$$

and the set

$$K_\lambda := \{x \in K : h(\lambda) = q(x) - \lambda ||Ax - b||^2\}.$$

The function h is finite valued and convex and so subdifferentiable with subdifferential

$$\partial h(\lambda) = \operatorname{conv}\{-||Ax_\lambda - b||^2 : x_\lambda \in K_\lambda\}.$$

The function h is based on the well known quadratic penalty function in nonlinear programming, see, e.g. [20]. Since q is a "nice" function and bounded below on K, the results follow from the well known results on this penalty function. Note that the penalty function exhibits monotonic behaviour, so that we get the exact penalty function behaviour on the finite set K, i.e., the existence of $\bar{\lambda}$. The fact that we do not need the nonnegativity restriction on λ follows by noting that in this case $-\lambda$ gives a smaller value for each x. Also, an optimal positive point, $\lambda > 0$, exists and $0 \in \partial h(\lambda)$. Since the elements in the subdifferential must be ≤ 0, this implies that there exists a feasible point x_λ which attains this minimum, i.e., this yields the result. □

4. Applications

In this section we study several specific instances of (P) and show how to apply the recipe for relaxations and perturbations. In each case we derive a min-max eigenvalue problem from the Lagrangian dual of an appropriately chosen quadratically constrained program. The dual of this dual problem provides a semidefinite relaxation for the original problem. We do this for: the quadratic assignment problem; graph partitioning; and the max-clique problem.

4.1. Quadratic Assignment Problem

Typical relaxations for QAP, see the definition in Section 1, try to exploit the trace formulation and use perturbations on A, B separately. Current approaches have two serious drawbacks. They completely discard the nonnegativity constraints and then they derive a bound from the sum of two bounds obtained by treating the quadratic and linear parts of the objective function separately, see, e.g. [22]. However, the Lagrangian relaxations and homogenization for the special case $S = \Re^n$ shows that we should consider more general perturbations and, in particular, we should consider perturbations that arise from Lagrangian quadratic relaxations. This approach does not have the two drawbacks mentioned above.

We now use the fact that the set of permutation matrices is equal to the intersection of the orthogonal matrices with the 0,1 matrices. We get the following equivalent program to QAP.

$$\begin{array}{rccc} \mu^* := & \max & q(X) = \operatorname{trace}(AXB - 2C)X^t & \\ (QAP_E) & \text{subject to} & XX^t = I & \\ & & X_{ij}^2 - X_{ij} = 0, \;\; \forall i,j. & \end{array} \qquad (4.27)$$

We could also consider the square of the norm of the residual of the (redundant) linear constraints

$$Xe = e, \;\; X^t e = e.$$

Other equivalent relaxations and bounds can be obtained by adding redundant constraints such as

$$\operatorname{trace} XX^t = n$$

or

$$0 \leq X_{ij} \leq 1, \ \forall i, j.$$

We now devote our attention to homogenization since that results in a min-max eigenvalue problem and an equivalent semidefinite programming problem. We have seen that we can homogenize by increasing the dimension of the problem by 1. We first add the 0,1 constraints to the objective function using Lagrange multipliers W_{ij}.

$$\min_{W} \max_{XX^t=I} \operatorname{trace}(AXB - 2C)X^t + \sum_{ij} W_{ij}(X_{ij}^2 - X_{ij}). \tag{4.28}$$

We now homogenize the objective function by multiplying by a constrained scalar x.

$$\min_{W} \max_{XX^t=I, x^2=1} \operatorname{trace} \left[AXBX^t + W(X \circ X)^t - x(2C + W)X^t \right]. \tag{4.29}$$

We can now use Lagrange multipliers to get a parametrized min-max eigenvalue problem in dimension $n^2 + 1$. We get the following bound. The parameters are: the symmetric $n \times n$ matrix $\Lambda = \Lambda^t$, the general $n \times n$ matrix W and the scalar α.

$$\begin{aligned} B_{QAP} := & \min_{\Lambda, W, \alpha} \max_{X} \operatorname{trace} [\\ & AXBX^t + \Lambda XX^t + W^t(X \circ X) + \alpha x^2 \\ & -x(2C + W)X^t \ ] - \alpha - \operatorname{trace} \Lambda. \end{aligned} \tag{4.30}$$

We have grouped the quadratic, original linear, and constant terms together. The hidden semidefinite constraint now yields a semidefinite programming problem.

$$\begin{array}{ll} \min & -\operatorname{trace} \Lambda - \alpha \\ \text{subject to} & L_Q + \operatorname{Arrow}(\alpha, \operatorname{vec}(W)) + \mathrm{B}^0\mathrm{Diag}(\Lambda) \preceq 0, \end{array} \tag{4.31}$$

where we define the matrix

$$L_Q := \begin{bmatrix} 0 & -\operatorname{vec}(C)^t \\ -\operatorname{vec}(C) & B \otimes A \end{bmatrix}, \tag{4.32}$$

and the linear operators

$$\operatorname{Arrow}(\alpha, \operatorname{vec}(W)) := \begin{bmatrix} \alpha & -\frac{1}{2}\operatorname{vec}(W)^t \\ -\frac{1}{2}\operatorname{vec}(W) & \operatorname{Diag}(\operatorname{vec}(W)) \end{bmatrix}, \tag{4.33}$$

$$\mathrm{B}^0\mathrm{Diag}(\Lambda) := \begin{bmatrix} 0 & 0 \\ 0 & I \otimes \Lambda \end{bmatrix}. \tag{4.34}$$

We can now introduce the $(n^2+1)\times(n^2+1)$ dual variable matrix $Y \succeq 0$ and derive the dual program to this min-max eigenvalue problem, i.e.

$$\max_{Y\succeq 0}\ \min_{\Lambda,W,\alpha}\ -\mathrm{trace}\,\Lambda - \alpha + \mathrm{trace}\,Y(L_Q + \mathrm{Arrow}\,(\alpha, \mathrm{vec}\,(W)) + \mathrm{B^0Diag}\,(\Lambda)).$$

The inner minimization problem is unconstrained and linear in the variables. Therefore, after reorganizing the variables, we can differentiate to get the dual problem to this dual problem, or the semidefinite relaxation to the original QAP. (Recall that $Y_{i,j:k}$ refers to the i-th row and columns j to k of the matrix Y; and $\mathrm{b^0diag}\,(Y)$ is the block diagonal sum of Y which ignores the first row.) The derivatives with respect to α and W yields the first constraint and the derivative with respect to Λ yields the second constraint in the following program. Equivalently, the constraints are the adjoints of the linear operators Arrow and $\mathrm{B^0Diag}$.

$$\begin{array}{rc} \max & \mathrm{trace}\,L_Q Y \\ \text{subject to} & \mathrm{diag}\,(Y) = (1, Y_{0,1:n^2})^t \\ & \mathrm{b^0diag}\,(Y) = I \\ & Y \succeq 0. \end{array} \tag{4.35}$$

Another primal-dual pair can be obtained using a trust region subproblem as the inner maximization problem, rather than homogenizing to an eigenvalue problem. This is done by adding the redundant trust region constraint trace $XX^t = n$. Also, as mentioned above, we can add the redundant constraint

$$||Xe - e||^2 + ||X^t e - e||^2 = 0.$$

This type of constraint is discussed below for the graph partitioning problem. A primal-dual interior point method based on the these types of dual pairs of programs, such as (4.35),(4.31), are being tested and studied in [14].

4.2. Graph Partitioning

Let $G = (V, E)$ be an undirected graph as in the description for (MC). The graph partitioning problem is the problem of partitioning the node set V into k disjoint subsets of specified sizes so as to minimize the total weight of the edges connecting nodes in distinct subsets of the partition. Let $A = (a_{ij})$ be the weighted adjacency matrix of G, i.e.

$$a_{ij} = \begin{cases} w_{ij} & ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

The graph partitioning problem can be described by the following (0,1)-quadratic program see, e.g. [24].

$$(\mathbf{GP}) \qquad \begin{array}{rc} w(E_{uncut}) = \max & \frac{1}{2}\mathrm{trace}\,X^tAX \\ \text{subject to} & Xe_k = e_n \\ & X^te_n = m \\ & X_{ij} \in \{0,1\},\ \forall ij, \end{array}$$

where e_k is the vector of ones of appropriate size and m is the vector of ordered set sizes

$$m_1 \geq \ldots \geq m_k \geq 1 \text{ and } k < n.$$

The columns of the 0,1 $n \times k$ matrices X are the indicator vectors for the sets. We can replace the 0,1 constraints by quadratics and also change the linear constraints to quadratic by squaring. We get the following equivalent program.

$$\begin{array}{rc} w(E_{uncut}) = \max & \frac{1}{2}\text{trace}\, X^t A X \\ \text{subject to} & \|Xe_k - e_n\|^2 + \|X^t e_n - m\|^2 = 0 \\ & X_{ij}^2 - X_{ij} = 0, \ \forall ij. \end{array}$$

The Lagrangian relaxation yields the following bound.

$$\begin{array}{rl} B_{GP} := & \min\limits_{\alpha, W} \max\limits_{X} \text{trace} \\ & [\ \frac{1}{2}X^t A X + \alpha(e_k e_k^t X^t X + X^t e_n e_n^t X) + W^t(X \circ X) \\ & -2\alpha(e_k e_n^t X + m e_n^t X) - W^t X\ ] \\ & +\alpha(n + \sum_i m_i^2). \end{array} \tag{4.36}$$

We can now homogenize the problem by adding a variable x.

$$\begin{array}{rl} B_{GP} := & \min\limits_{\alpha, W} \max\limits_{\substack{X \\ x^2=1}} \text{trace} \\ & [\ \frac{1}{2}X^t A X + \alpha(e_k e_k^t X^t X + X^t e_n e_n^t X) + W^t(X \circ X) \\ & +x(-2\alpha(e_k e_n^t X + m e_n^t X) - W^t X)\ ] \\ & +\alpha(n + \sum_i m_i^2). \end{array}$$

We now lift the variable x into the Lagrangian to get a min-max eigenvalue problem.

$$\begin{array}{rl} B_{GP} := & \min\limits_{\alpha, W, \delta} \max\limits_{X, x} \text{trace} \\ & [\ \frac{1}{2}X^t A X + \alpha(e_k e_k^t X^t X + X^t e_n e_n^t X) + W^t(X \circ X) + \delta x^2 \\ & +x(-2\alpha(e_k e_n^t X + m e_n^t X) - W^t X)\ ] \\ & +\alpha(n + \sum_i m_i^2) - \delta. \end{array}$$

The above has a hidden semidefinite constraint.

$$\begin{array}{rc} \min & \alpha(n + \sum_i m_i^2) - \delta \\ \text{subject to} & L_A + \text{Arrow}(\delta, \text{vec}(W)) + \alpha L_\alpha \preceq 0, \end{array} \tag{4.37}$$

where we define the matrices

$$L_A := \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}I \otimes A \end{bmatrix}, \tag{4.38}$$

$$v = \text{vec}\, e_n m^t,$$

$$L_\alpha := \begin{bmatrix} 0 & -(e+v)^t \\ -(e+v) & (e_k e_k^t I \otimes I + I \otimes e_n e_n^t) \end{bmatrix}, \tag{4.39}$$

and the linear operator

$$\text{Arrow}\,(\delta, \text{vec}\,(W)) := \begin{bmatrix} \delta & -\frac{1}{2}(\text{vec}\,(W))^t \\ -\frac{1}{2}(\text{vec}\,(W)) & \text{Diag}\,(\text{vec}\,(W)) \end{bmatrix}. \tag{4.40}$$

The dual program yields the semidefinite relaxation of (GP).

$$\begin{array}{ll} \max & \text{trace}\, L_A Y \\ \text{subject to} & \text{diag}\,(Y) = (1, Y_{0,1:n})^t \\ & \text{trace}\, Y L_\alpha = 0 \\ & Y \succeq 0. \end{array} \tag{4.41}$$

4.3. MAX-CLIQUE AND STABLE SET

Consider again the undirected graph $G = (E, V)$ defined above. The *max-clique* problem consists in finding the largest connected subgraph. We let $\omega(G)$ denote the size of the largest clique in G. A *stable set* is a subset of vertices of V such that no two vertices are adjacent. We denote the size of the largest stable set in $\bar{G}$, the complement of G, by $\alpha(\bar{G})$. Clearly

$$\alpha(\bar{G}) = \omega(G).$$

Bounds for these problems and relationships to the theta function, or Lovász number of the graph, are described in the expository paper, e.g. [15]; see also [25].

In this section we show that the Lovasz bound on $\omega(G)$ can be alternatively obtained from two distinct 01-programs (4.42) and (4.45) by Lagrangian relaxations. Let A be the incidence matrix of the graph, i.e. $A = (a_{ij})$ with $a_{ij} = 1$ if $ij \in E$ and 0 otherwise. If x is the indicator vector for the largest clique in G of size k, A then $x^t(I+A)x/x^t x = k^2/k = k$. A quadratic formulation of the max-clique problem is the following (0,1)-quadratic program.

$$\begin{array}{lll} \omega(G) = & \max & \frac{x^t(I+A)x}{x^t x} \\ & \text{subject to} & x_i x_j = 0, \text{ if } ij \notin E,\ i \neq j \\ & & x_i \in \{0,1\},\ \forall i. \end{array} \tag{4.42}$$

Therefore, a quadratic relaxation of the max-clique problem is the following quadratic constrained program.

$$\begin{array}{lll} \omega(G) \le \omega_1^* := & \max & x^t(I+A)x \\ & \text{subject to} & x_i x_j = 0, \text{ if } ij \notin E,\ i \neq j \\ & & x^t x = 1. \end{array} \tag{4.43}$$

The Lagrangian relaxation for this problem is the perturbed min-max eigenvalue problem and the equivalent semidefinite program

$$\begin{aligned}
\omega_1^* &\leq \min_{W_{ij}=0,\text{ if } ij\in E,\text{ or } i=j} \lambda_{\max}(I+A+W) - \alpha x^t x + \alpha \\
&= \min_{w,\alpha}\max_{x} x^t(I+A)x + \sum_{ij\notin E,\, i\neq j} w_{ij}x_ix_j - \alpha x^t x + \alpha \\
&= \min_{\substack{I+A+W\preceq \alpha I \\ W_{ij}=0,\text{ if } ij\in E,\text{ or } i=j}} \alpha
\end{aligned}$$

i.e. minimize the max eigenvalue over perturbations in the off-diagonal elements corresponding to disjoint nodes. This bound is equal to the Lovasz theta function on the complementary graph.

$$\vartheta(\bar{G}) = \min_{A\in\mathcal{A}} \lambda_{\max}(A), \tag{4.44}$$

where $\mathcal{A} = \{A : A$ symmetric $n \times n$ matrix with $A_{ij} = 1$, if $ij \in E$, or $i = j\}$.

By considering the (optimal) indicator vector for the largest clique, we see that a (0,1)-quadratic program that describes the max-clique problem exactly is the following one. Note that if node i is not in the largest clique, then necessarily, $x_ix_j = 0$ for some j with node j in the clique, i.e. necessarily $x_i = 0$ in the indicator vector.

$$\begin{aligned}
\omega(G) = \quad & \max \quad x^t x \\
& \text{subject to} \quad x_ix_j = 0, \text{ if } ij \notin E,\ i\neq j \\
& \qquad\qquad\quad x_i^2 - x_i = 0,\ \forall i.
\end{aligned} \tag{4.45}$$

The Lagrangian relaxation yields the bound

$$B_{\text{clique}} := \min_{W,\lambda}\max_{x} x^t x + \sum_{ij\notin E,\, i\neq j} w_{ij}x_ix_j + \sum_i \lambda_i(x_i^2 - x_i).$$

We let W be an $n \times n$ matrix with zeros in positions where $ij \in E$. We can homogenize by adding the constraint $y^2 = 1$ and then lifting it into the Lagrangian.

$$\min_{\alpha,W,\lambda}\max_{x,y} x^t x + \sum_{ij\notin E} w_{ij}x_ix_j + \sum_i \lambda_i x_i^2 + \alpha y^2 - y\sum_i \lambda_i x_i - \alpha.$$

We now exploit the hidden semidefinite constraint to get the semidefinite program.

$$\begin{aligned}
B_{\text{clique}} = \quad & \min_{W,\lambda,\alpha} \quad -\alpha \\
& \text{subject to} \quad L_A + L_W(W) + \text{Arrow}(\alpha,\lambda) \preceq 0 \\
& \qquad\qquad\quad W_{ij} = 0,\ \forall ij \in E,\text{ or } i = j,
\end{aligned} \tag{4.46}$$

where the matrix

$$L_A := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \tag{4.47}$$

and the linear operators

$$L_W(W) := \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix}, \tag{4.48}$$

$$\mathrm{Arrow}\,(\alpha, \lambda) := \begin{bmatrix} \alpha & -\frac{1}{2}\lambda^t \\ -\frac{1}{2}\lambda & \mathrm{Diag}\,(\lambda) \end{bmatrix}. \tag{4.49}$$

The dual of the above min-max eigenvalue problem yields the semidefinite relaxation for the max-clique problem with $Y \in \mathcal{S}_{n+1}$.

$$\begin{array}{rl} \max & \mathrm{trace}\, L_A Y \\ \text{subject to} & \mathrm{diag}\,(Y) = (1, Y_{0,1:n})^t \\ & Y_{ij} = 0, \ \forall ij \notin E \\ & Y \succeq 0. \end{array} \tag{4.50}$$

The equivalence of the bounds (4.44) and (4.50) was shown in lemma 2.17 of [19].

Consider the program (4.42) with an additional redundant constraint

$$x_i x_j \geq 0 \text{ for } ij \in E \tag{4.51}$$

That is

$$\begin{array}{rrl} \omega(G) = & \max & \frac{x^t(I+A)x}{x^t x} \\ & \text{subject to} & x_i x_j = 0, \ \text{if } ij \notin E, \ i \neq j \\ & & x_i x_j \geq 0, \ \text{if } ij \in E, \\ & & x_i \in \{0, 1\}, \ \forall i. \end{array} \tag{4.52}$$

A quadratic relaxation of the max-clique problem is the following quadratic constrained program.

$$\begin{array}{rrl} \omega(G) \leq \omega_1^* := & \max & x^t(I+A)x \\ & \text{subject to} & x_i x_j = 0, \ \text{if } ij \notin E, \ i \neq j \\ & & x_i x_j \geq 0, \ \text{if } ij \in E, \\ & & x^t x = 1. \end{array} \tag{4.53}$$

The Lagrangian relaxation for this problem is equal to the Schrijver's improvement [25] of the theta function on the complementary graph.

$$\vartheta'(\bar{G}) = \min_{A \in \mathcal{A}'} \lambda_{\max}(A),$$

where $\mathcal{A}' = \{A : A \text{ symmetric } n \times n \text{ matrix with } A_{ij} \geq 1, \text{ if } ij \in E, \text{ or } i = j\}$. Haemmers [11] constructed graphs where $\vartheta'(\bar{G})$ is strictly smaller than $\vartheta(\bar{G})$.

Analogously, it is possible to modify the program (4.45) by adding the constraint (4.51).

5. Concluding Remarks

We have found the best quadratic relaxation, among all quadratic majorants with zero trace, for constrained (0,1)-quadratic programming; and we have shown that this is equal to a semidefinite relaxation. We have also provided a recipe for calculating a dual pair of semidefinite programs for primal-dual algorithms. This dual pair of programs are suitable for interior point semidefinite programming techniques.

In particular, we have provided a primal-dual pair of semidefinite programs for the quadratic assignment problem, graph partitioning problem, and the max-clique problem. The semidefinite relaxations for these problems are interesting for both numerical and theoretical reasons. The feasible sets have empty interior and so a relative interior primal-dual method must be used. Numerical tests are currently being studied in [14, 7].

Acknowledgements

We would like to thank an anonymous referee for his help in improving the original version of this paper.

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